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Quantum Mechanics of Gravitational Waves

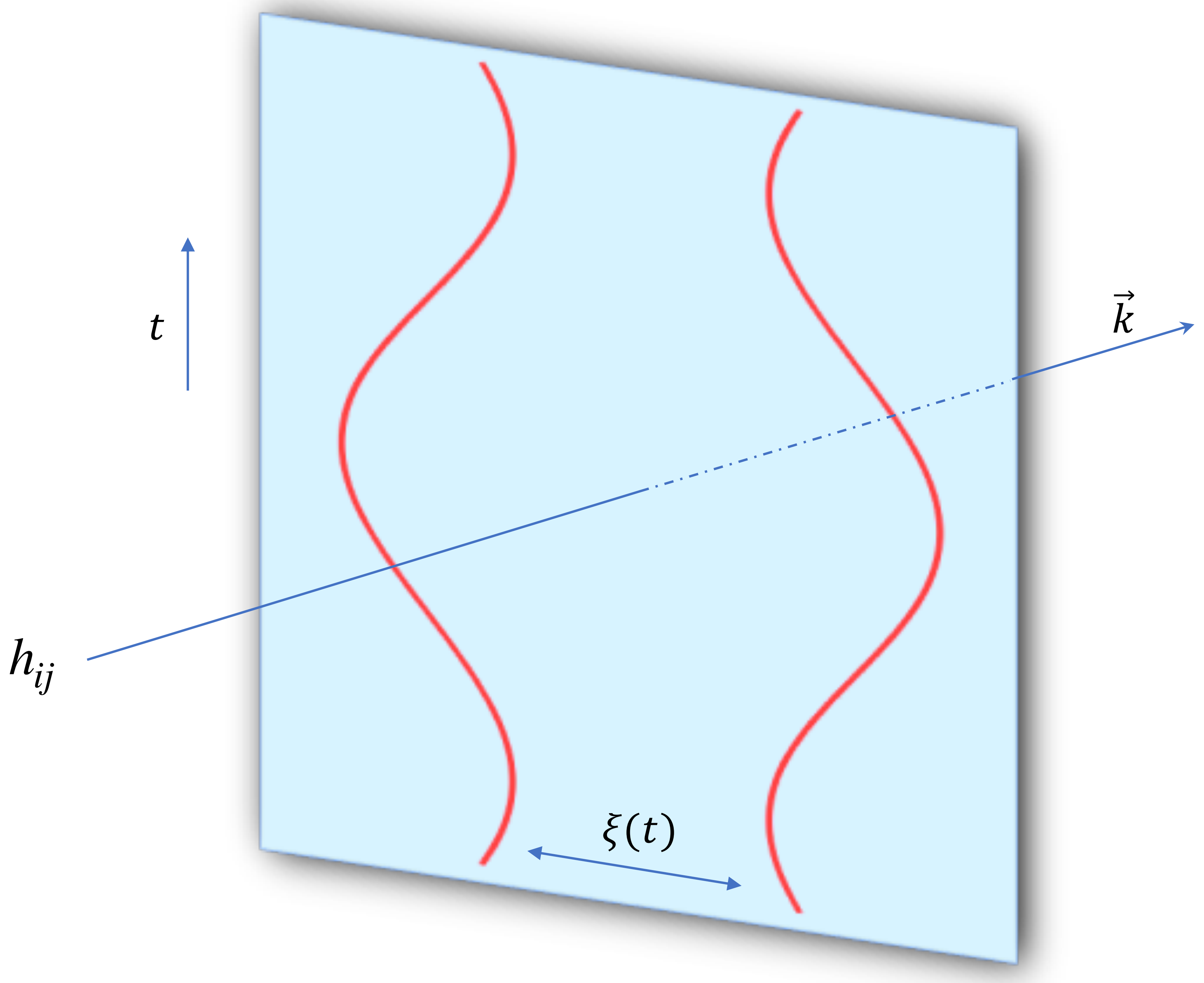
George Zahariade

Institut de Física d'Altes Energies

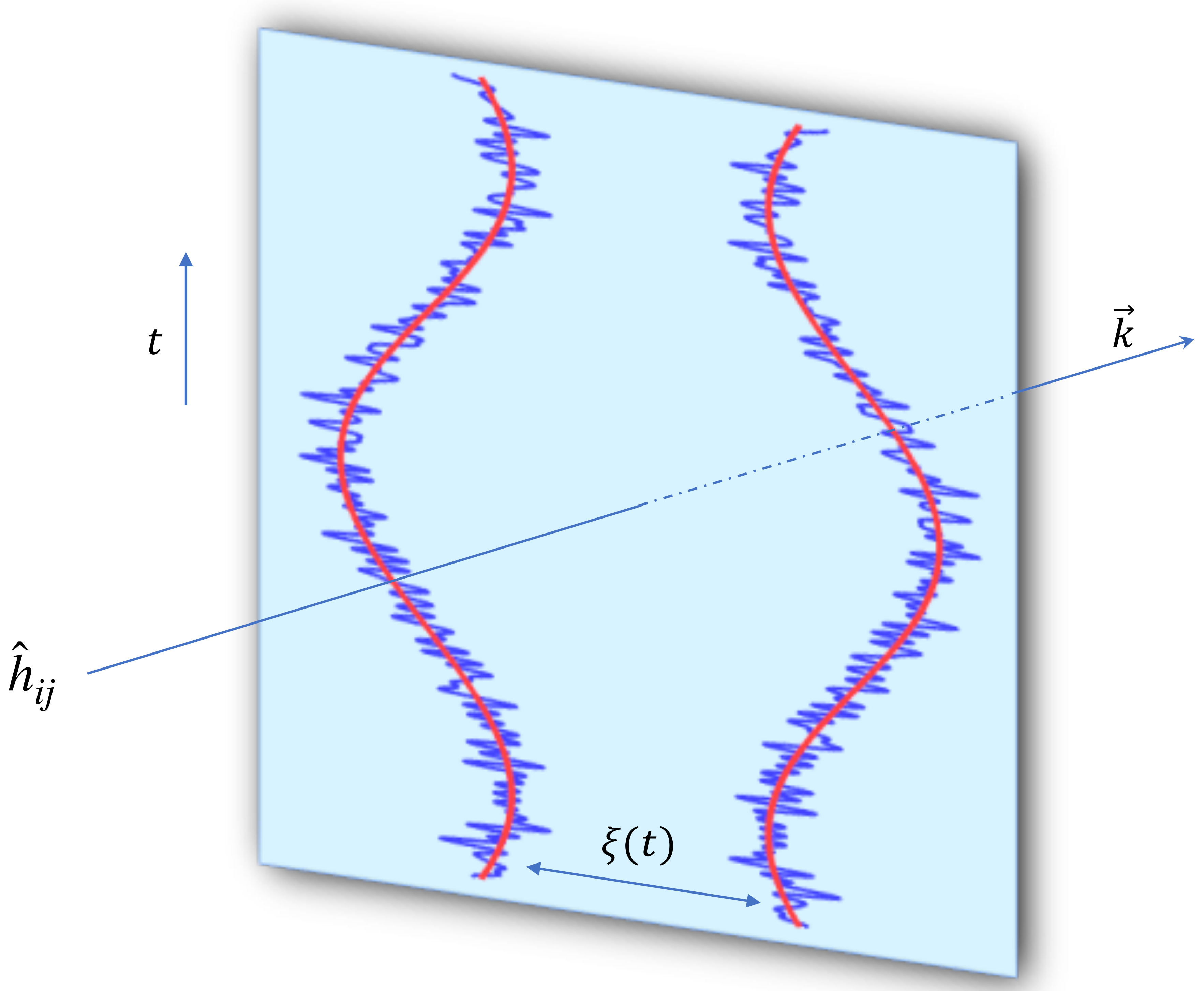
References

- Papers on which this talk is based:
  - ***M. Parikh, F. Wilczek, GZ,*** "The Noise of Gravitons"
    arXiv:2005.07211
  - ***M. Parikh, F. Wilczek, GZ,*** "Quantum Mechanics of Gravitational Waves"
    arXiv:2010.08205
  - ***M. Parikh, F. Wilczek, GZ,*** "Signatures of the Quantization of Gravity at Gravitational Wave Detectors"
    arXiv:2010.08208
- Inspired by the seminal work of *Feynman* and *Vernon*
  "The Theory of a general quantum system interacting with a linear dissipative system" (1963)
- And subsequent work by *Hu, Calzetta*… about open quantum systems and stochastic gravity
- Topic studied by other groups: ***Kanno, Soda, Tokuda*** (2020-21), ***Kanno, Soda*** (2021), *Hertzberg, Litterer* (2021), *Guerreiro, Frassino et al.* (2020-2022)

Basic Idea



Basic Idea



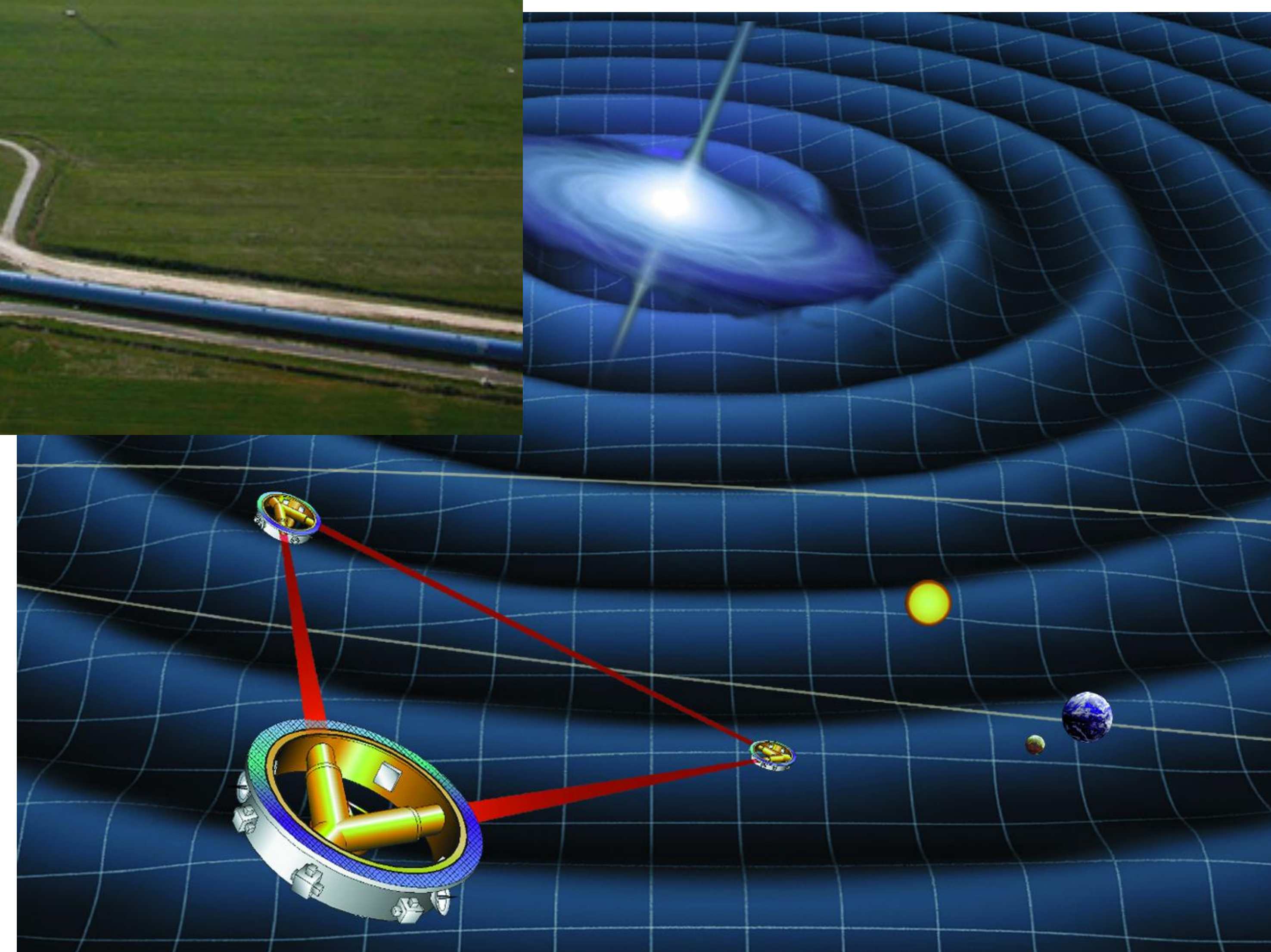
Outline

- **Detector Model:** arm length of GW interferometer coupled to weak gravity
- **Quantization of Weak Gravitational Field:** Feynman-Vernon influence functional
- **Effective Dynamics for GW Detector:** Langevin-like equation
- **Noise Characteristics** for different quantum states of GW

Detector Model

Credit: The Virgo Collaboration/CCO 1.0

Credit: NASA/Public Domain



Interferometer arm

$\approx$ 2 freely falling masses

Detector Model

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - M\int d\lambda\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} - m\int d\tau\sqrt{-g_{\mu\nu}\frac{dY^\mu}{d\tau}\frac{dY^\nu}{d\tau}}$$

Einstein-Hilbert action

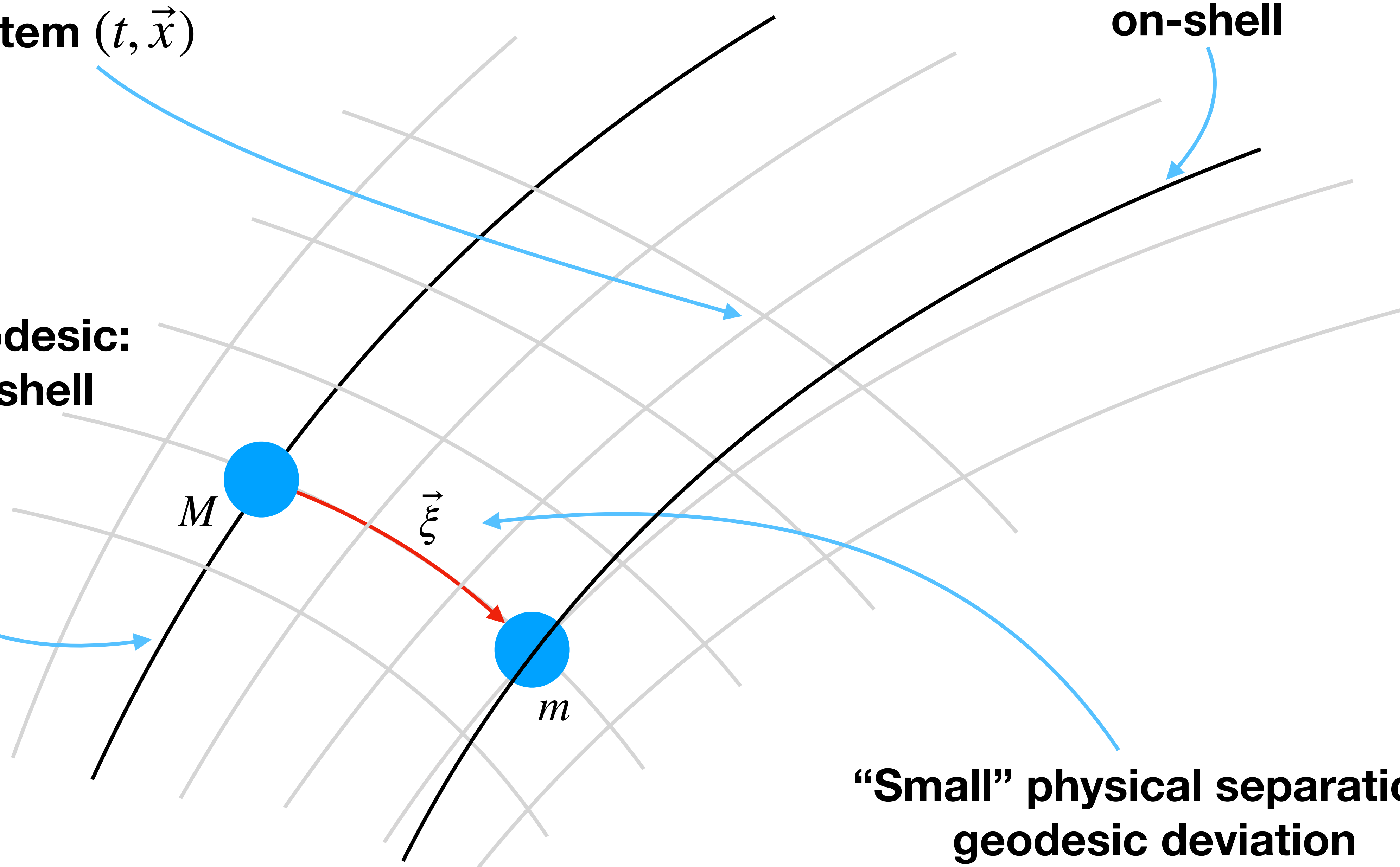
2 free particles

Detector Model

TT gauge coordinate system $(t, \vec{x})$

Light m **not necessarily on-shell**

Reference geodesic: Heavy M **on-shell**



“Small” physical separation: geodesic deviation

$$\ddot{\xi}(t) - \frac{1}{2}\ddot{h}(t)\xi(t) = 0$$

Detector Model

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - M\int d\lambda\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} - m\int d\tau\sqrt{-g_{\mu\nu}\frac{dY^\mu}{d\tau}\frac{dY^\nu}{d\tau}}$$

Einstein-Hilbert action **2 free particles**

- $M \gg m$ **:** heavy particle on-shell (geodesic motion)

- **Fermi normal coordinates:** $X^\mu = (t, \vec{0})$ and $Y^\mu = (t, \vec{\xi})$

$$g_{00}(t,\xi) = -1 - R_{i0j0}(t,0)\xi^i\xi^j + O(\xi^3)$$

$$g_{0i}(t,\xi) = O(\xi^2)$$

$$g_{ij}(t,\xi) = \delta_{ij} + O(\xi^2)$$

Detector Model

- **Weak gravity:** expand at quadratic order in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$; TT gauge choice

- **Small physical separation:** quadrupole approximation; non-relativistic limit; expand at quadratic order in ξ

- **Keep lowest interacting order**

$$\boxed{S = -\frac{1}{64\pi G}\int d^4x\,\partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt\left(\frac{1}{2}m\dot{\xi}^2 + \frac{1}{4}m\ddot{h}_{ij}(t,0)\xi^i\xi^j\right)}$$

Detector Model

- **Weak gravity:** expand at quadratic order in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$; TT gauge choice

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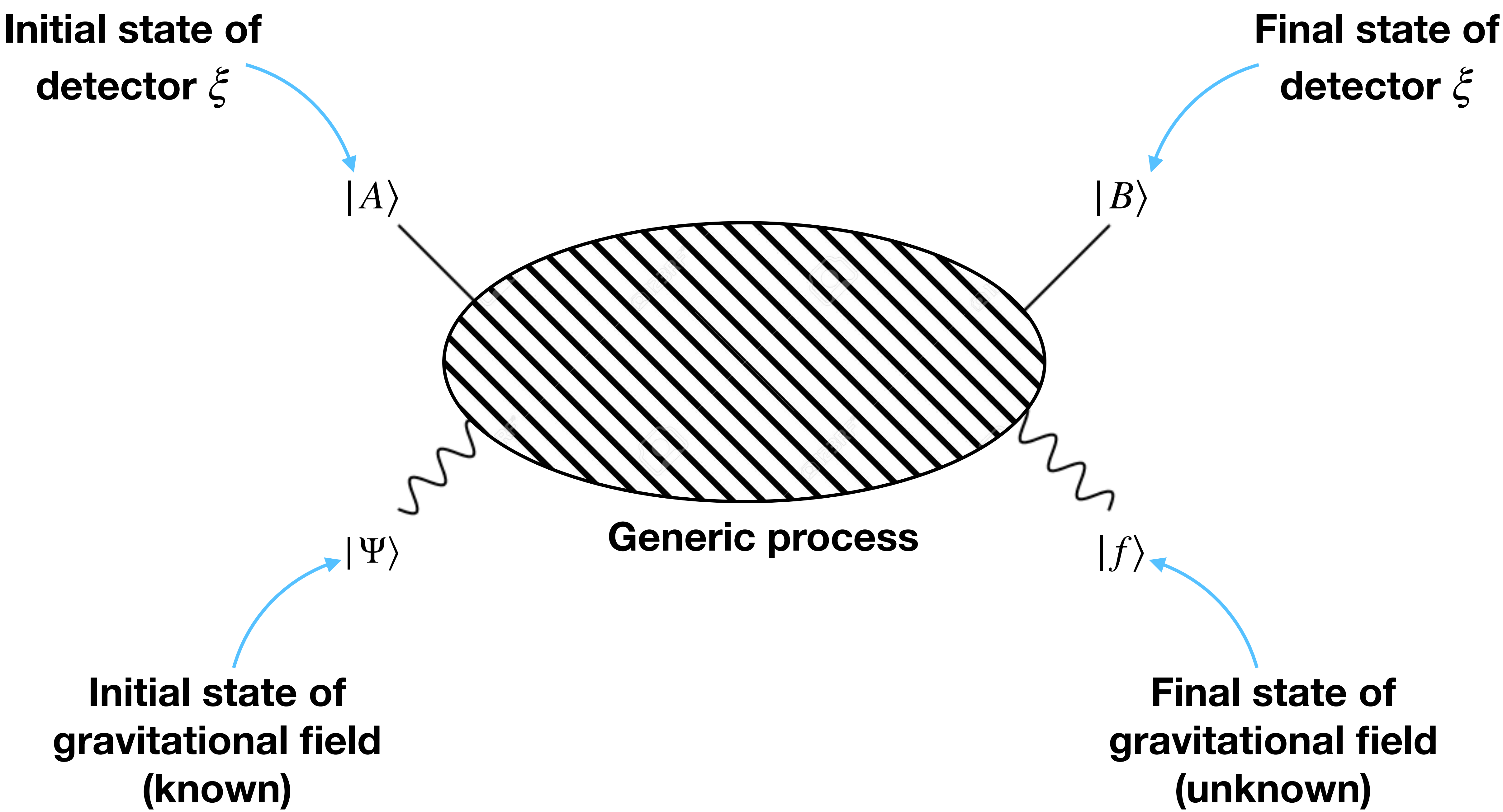
$$\boxed{S = -\frac{1}{64\pi G}\int d^4x\, \partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt\left(\frac{1}{2}m\dot{\xi}^2 + \frac{1}{4}m\ddot{h}_{ij}(t,0)\xi^i\xi^j\right)}$$

$\propto R_{0i0j}$

Detector response to quantized GW



Detector response to quantized GW

- **Quantity of interest:** transition probability from $|A\rangle$ to $|B\rangle$ given incoming gravitational field state $|\psi\rangle$ (in time T)

- **Final state $|f\rangle$ unknown:** sum over final states

$$P_\Psi(A \to B) = \sum_{|f\rangle} |\langle f, B | \hat{U}(T) | \Psi, A\rangle|^2$$

Evaluated as a path integral $\sim \int \mathscr{D}\xi \int \mathscr{D}h \, e^{\frac{i}{\hbar}S}$

$$\left(S = -\frac{1}{64\pi G}\int d^4x \, (\partial h)^2 + \int dt \left(\frac{1}{2}m\dot{\xi}^2 + \frac{1}{4}m\ddot{h}(t,0)\xi^2 \right) \right)$$

Influence Functional

$$P_\Psi(A \to B) \sim \int \mathscr{D}\xi\mathscr{D}\xi' \exp\left[\frac{i}{\hbar}\int_0^T dt\frac{1}{2}m(\dot{\xi}^2 - \dot{\xi}'^2)\right]$$

$$\times \sum_{|f\rangle}\int \mathscr{D}h\mathscr{D}h' \exp\left[-\frac{i}{64\pi G\hbar}\int d^4x \left((\partial h)^2 - (\partial h')^2\right)\right.$$

$$\left.+\frac{i}{\hbar}\int dt\frac{1}{4}m\left(\ddot{h}(t,0)\xi^2 - \ddot{h}'(t,0)\xi'^2\right)\right]$$

Influence Functional

$$P_\Psi(A \to B) \sim \int \mathscr{D}\xi \mathscr{D}\xi' \exp\left[\frac{i}{\hbar}\int_0^T dt \frac{1}{2} m(\dot{\xi}^2 - \dot{\xi}'^2)\right]$$

$$\times \sum_{|f\rangle} \int \mathscr{D}h \mathscr{D}h' \exp\left[-\frac{i}{64\pi G\hbar}\int d^4x \left((\partial h)^2 - (\partial h')^2\right)\right.$$

$$\left. + \frac{i}{\hbar}\int dt \frac{1}{4} m \left(\ddot{h}(t,0)\xi^2 - \ddot{h}'(t,0)\xi'^2\right)\right]$$

Boundary conditions depend on $|\Psi\rangle$ and $|f\rangle$

Gravitational part of the action: quadratic in h_{ij}

Influence Functional

$$P_\Psi(A \to B) \sim \int \mathscr{D}\xi \mathscr{D}\xi' \exp\left[\frac{i}{\hbar}\int_0^T dt \frac{1}{2} m(\dot{\xi}^2 - \dot{\xi}'^2)\right]$$

$$\times \boxed{\sum_{|f\rangle} \int \mathscr{D}h \mathscr{D}h' \exp\left[-\frac{i}{64\pi G\hbar}\int d^4x \, \left((\partial h)^2 - (\partial h')^2\right) + \frac{i}{\hbar}\int dt \frac{1}{4} m \left(\ddot{h}(t,0)\xi^2 - \ddot{h}'(t,0)\xi'^2\right)\right]}$$

$F_\Psi[\xi, \xi']$: **encodes the effects of the quantum fluctuations of h_{ij} on ξ**

Analysis of the Influence Functional

- ***Choice* of the quantum state $|\Psi\rangle$:** state corresponding to a classical gravitational wave profile $h_{\text{cl}}(t)$

- **Field coherent state:** $|\Psi\rangle = \bigotimes_{\omega} |\psi_{\omega}\rangle = \bigotimes_{\omega} \hat{D}(\alpha_{\omega})\,|0_{\omega}\rangle$

  where $\alpha_{\omega} \propto \int dt\ h_{cl}(t) e^{-i\omega t}$

- **Factorized influence functional:**

$$F_{\Psi}[\xi, \xi'] = F_{0}[\xi, \xi']\, e^{\frac{i}{\hbar}\int_0^T dt \frac{m}{4}\ddot{h}_{\text{cl}}(t)(\xi^2 - \xi'^2)}$$

Vacuum influence functional

Analysis of the vacuum Influence Functional

- **Dissipation term**

$$\mathrm{Arg}\, F_0 = -\frac{m^2 G}{8\hbar}\int_0^T dt\ \big(X(t) - X'(t)\big)\ \big(\dot{X}(t) + \dot{X}'(t)\big) \qquad \left(X = \frac{d^2}{dt^2}\xi^2\right)$$

- **Fluctuation term**

$$|F_0| = exp\left[-\frac{m^2}{32\hbar^2}\int_0^T\int_0^T dt\, dt'\ A(t-t')\big(X(t) - X'(t)\big)\ \big(X(t') - X'(t')\big)\right]$$

Singular function that is exactly computable

Analysis of the Influence Functional

$$\exp\left[-\frac{m^2}{32\hbar^2}\int_0^T\int_0^T dt\,dt'\,A(t-t')\big(X(t)-X'(t)\big)\,\big(X(t')-X'(t')\big)\right] =$$

$$\int \mathscr{D}N \exp\left[-\frac{1}{2}\int_0^T\int_0^T dt\,dt'\,A^{-1}(t-t')N(t)N(t') + \frac{i}{\hbar}\int_0^T dt\frac{m}{4}N(t)\big(X(t)-X'(t)\big)\right]$$

$N(t)$**: zero-mean Gaussian stochastic function**
with auto-correlation $A(t-t')$ **and power spectrum** $S(\omega)$

Analysis of the vacuum Influence Functional

- **Dissipation term**

$$\mathrm{Arg}\, F_0 = -\frac{m^2 G}{8\hbar}\int_0^T dt\ \big(X(t) - X'(t)\big)\ \big(\dot{X}(t) + \dot{X}'(t)\big) \qquad \left(X = \frac{d^2}{dt^2}\xi^2\right)$$

- **Fluctuation term**

$$|F_0| = \left\langle \exp\left(\frac{i}{\hbar}\int_0^T dt \frac{m}{4} N(t)\big(X(t) - X'(t)\big)\right)\right\rangle_N$$

Stochastic average over Gaussian "noise" $N(t)$

Back to the Transition Probability

Gaussian distribution

$$
\begin{aligned}
P_\Psi(A \to B) \sim \int \mathscr{D}\xi \mathscr{D}\xi' \mathscr{D}N \exp\left[-\frac{1}{2}\int_0^T \int_0^T dt\, dt'\, A^{-1}(t-t')N(t)N(t')\right] \times \\
\exp\Bigg[\frac{i}{\hbar}\int_0^T dt \left\{\frac{1}{2}m\left(\dot{\xi}^2 - \dot{\xi}'^2\right) + \frac{m}{4}\ddot{h}_{\mathrm{cl}}(t)\left(\xi^2(t) - \xi'^2(t)\right)\right\} \\
-\frac{im^2G}{8\hbar}\int_0^T dt\, \left(X(t) - X'(t)\right)\left(\dot{X}(t) + \dot{X}'(t)\right) \\
+\frac{i}{\hbar}\int_0^T dt \frac{m}{4} N(t)\left(X(t) - X'(t)\right)\Bigg]
\end{aligned}
$$

Classical piece

Dissipation term

Fluctuation term

Langevin equation

Stationary phase approximation:
stochastic equation for the detector

$$\boxed{\ddot{\xi}(t) - \frac{1}{2}\left[\ddot{h}_{\mathrm{cl}}(t) + \ddot{N}(t) - \frac{mG}{c^5}\frac{d^5}{dt^5}\xi^2(t)\right]\xi(t) = 0}$$

Classical wave profile

Vacuum fluctuations

Radiation reaction

Effective equation of motion for the detector including quantum effects

Analysis of Noise

Equilibrium arm length
$\sim 1\text{km} - 10^6\text{km}$

Cutoff set by sensitivity of detector
$\sim 1\text{rad . s}^{-1} - 10^6\text{rad . s}^{-1}$

- **Estimate of noise:** $\sigma^2 \sim \xi_0^2 \int_0^{\omega_{\max}} d\omega S(\omega)$
- **Vacuum and coherent states:** $S(\omega) = 4G\hbar\omega/c^5$
- **Thermal states:** $S(\omega) = \dfrac{4G\hbar\omega}{c^5}\coth\left(\dfrac{\hbar\omega}{2k_BT}\right)$
- **Squeezed states:** $S(\omega) = 4e^r G\hbar\omega/c^5$

Exponential enhancement

Analysis of Noise

Equilibrium arm length
$\sim 1\text{km} - 10^6\text{km}$

Cutoff set by sensitivity of detector
$\sim 1\text{rad}\,.\,\text{s}^{-1} - 10^6\text{rad}\,.\,\text{s}^{-1}$

- **Estimate of noise:** $\sigma^2 \sim \xi_0^2 \int_0^{\omega_{\max}} d\omega S(\omega)$
- **Vacuum and coherent states:** tiny despite claims
  $\sigma_0 \sim \ell_P \xi_0 \omega_{\max}/c \lesssim 10^{-35}\text{m}$
- **Thermal states:** $\sigma \sim \sigma_0 \sqrt{k_B T/\hbar\omega_{\max}} \lesssim 10^{-28} - 10^{-31}\text{m}$
- **Squeezed states:** $\sigma \sim e^{r/2}\sigma_0$

Cosmic background (evaporating BHs?)

Cosmology/non-linear effects in binary BH mergers

Exponential enhancement
r **<** ***O*****(100) (*****Hetzberg&Litterer*** **2021)**

Summary

- **Model GW detector:** cubic interaction $h\xi^2$ (truncation)
- **Stochastic equation:** non-linear Langevin equation
- **Fundamental noise:** tiny BUT potentially enhanced for non-coherent states
- **Influence Functional:** semi-classical limit, radiation reaction
- **Open questions:** estimate squeezing, precise accounting of detector characteristics, use influence functionals to study backreaction…

Computation of the Influence Functional

$$P_\Psi(A \to B) \sim \int \mathscr{D}\xi \mathscr{D}\xi' e^{\frac{i}{\hbar}\int_0^T dt \frac{1}{2} m(\dot{\xi}^2 - \dot{\xi}'^2)} F_\Psi[\xi, \xi']$$

Encodes all the quantum effects of h_{ij} on ξ

$$F_\Psi[\xi, \xi'] = \langle \Psi \,|\, U^\dagger_{\xi'}(T) U_\xi(T) \,|\, \Psi \rangle$$

Time evolution operators (gravitational part of the action)

Computation of the Influence Functional

- **First step:** mode decomposition

$$S_{h,\xi} = -\frac{1}{64\pi G}\int d^4x\,\partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt\frac{1}{4}m\ddot{h}_{ij}(t,0)\xi^i\xi^j$$

$$= \int dt \sum_{\vec{k},s}\left[\frac{1}{2}\dot{h}^2_{\vec{k},s} - \frac{1}{2}\omega^2_{\vec{k}}h^2_{\vec{k},s} + \frac{1}{4}g\,m\ddot{h}_{\vec{k},s}\epsilon^s_{ij}(\vec{k})\xi^i\xi^j\right]$$

Coupling constant involving $G, \hbar$

Polarization

- **Simplification:** one direction, orthogonal to $\vec{\xi}$, single-polarization

$$S_{h,\xi} = \sum_\omega\left[\int dt\left(\frac{1}{2}\dot{h}^2_\omega - \frac{1}{2}\omega^2 h^2_\omega\right) + \int dt\frac{1}{4}g\,m\ddot{h}_\omega\xi^2\right]$$

Computation of the Influence Functional

- **Second step:** mode by mode quantization $|\Psi\rangle = \bigotimes_{\omega} |\psi_\omega\rangle$

$$\hat{H}_\xi = \hat{H}_{SHO} + \hat{H}_\xi^{\text{int}}$$

Free hamiltonian common to all modes

$\propto (\hat{a} + \hat{a}^\dagger)\xi^2$

- **Third step:** interaction picture + BCH formula

$$F_\Psi[\xi, \xi'] = F_0[\xi, \xi'] \prod_{\omega} \langle \psi_\omega | e^{-W^* \hat{a}^\dagger} e^{W \hat{a}} | \psi_\omega \rangle$$

Known functionals of ξ